

# A NOTE ON $p$ -ADIC $q$ -INTEGRAL ASSOCIATED WITH $q$ -EULER NUMBERS

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ABSTRACT. We show that  $q$ -Euler numbers can be represented as an integral by the  $q$ -analogue of the ordinary  $p$ -adic fermionic measure, whence we give an answer to the question of readers to ask us for the fermionic  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$ . Sometimes several readers give the nonsense comments to us. But we do not write their names in this paper. I would like to tell the readers that papers are different from books. That is, the readers are need their efforts to understand the contents of paper.

## §1. Introduction

Throughout this paper  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{Z}_p$ ,  $\mathbb{Q}_p$  and  $\mathbb{C}_p$  will respectively denote the ring of rational integers, the field of rational numbers, the ring  $p$ -adic rational integers, the field of  $p$ -adic rational numbers and the completion of the algebraic closure of  $\mathbb{Q}_p$ . Let  $v_p$  be the normalized exponential valuation of  $\mathbb{C}_p$  such that  $|p|_p = p^{-v_p(p)} = p^{-1}$ . If  $q \in \mathbb{C}_p$ , we normally assume  $|q - 1|_p < p^{-\frac{1}{p-1}}$ , so that  $q^x = \exp(x \log q)$  for  $|x|_p \leq 1$ . We use the notation

$$[x]_q = \frac{1 - q^x}{1 - q}, \text{ and } [x]_{-q} = \frac{1 - (-q)^x}{1 + q}.$$

Hence,  $\lim_{q \rightarrow 1} [x]_q = 1$ , for any  $x$  with  $|x|_p \leq 1$  in the present  $p$ -adic case.

Let  $p$  be a fixed odd prime. For  $d(= \text{odd})$  a fixed positive integer with  $(p, d) = 1$ , let

$$\begin{aligned} X &= X_d = \varprojlim_N \mathbb{Z}/dp^N \mathbb{Z}, \\ X_1 &= \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0 < a < dp \\ (a, p) = 1}} (a + dp\mathbb{Z}_p), \\ a + dp^N \mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned}$$

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where  $a \in \mathbb{Z}$  lies in  $0 \leq a < dp^N$ .

In this paper we prove that

$$\mu_{-q}(a + dp^N \mathbb{Z}_p) = (1 + q) \frac{(-1)^a q^a}{1 + q^{dp^N}} = \frac{(-q)^a}{[dp^N]_{-q}},$$

is distribution on  $X$  for  $q \in \mathbb{C}_p$  with  $|1 - q|_p < p^{-\frac{1}{p-1}}$ . This distribution yields an integral as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x) (-q)^x, \text{ for } f \in UD(\mathbb{Z}_p),$$

which has a sense as we see readily that the limit is convergent.

For  $q = 1$ , we have fermionic  $p$ -adic integral on  $\mathbb{Z}_p$  as follows:

$$I_{-1} = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^x.$$

In view of notation,  $I_{-1}$  can be written symbolically as  $I_{-1}(f) = \lim_{q \rightarrow -1} I_q(f)$ . Finally, we will introduce  $q$ -extension of Euler numbers by using  $I_{-q}(f)$

## §2. A note on fermionic $p$ -adic $q$ -integral on $\mathbb{Z}_p$

For any positive integer  $N$ ,  $q \in \mathbb{C}_p$  with  $|1 - q|_p < p^{-\frac{1}{p-1}}$ , we set

$$(*) \quad \mu_{-q}(a + dp^N \mathbb{Z}_p) = (1 + q) \frac{(-1)^a q^a}{1 + q^{dp^N}} = \frac{(-q)^a}{[dp^N]_{-q}},$$

and this can be extended to a distribution on  $X$ .

We show that  $\mu_{-q}$  is distribution on  $X$ . For this it suffices to check that

$$\sum_{i=0}^{p-1} \mu_{-q}(a + idp^N + dp^{N+1} \mathbb{Z}_p) = \mu_{-q}(a + dp^N \mathbb{Z}_p).$$

The left hand side is equal to

$$(1) \quad (1 + q) \sum_{i=0}^{p-1} \frac{(-1)^{a+idp^N}}{1 + q^{dp^{N+1}}} q^{a+idp^N} = (1 + q) \frac{(-1)^a q^a}{1 + q^{dp^{N+1}}} \sum_{i=0}^{p-1} q^{idp^N} (-1)^i.$$

Since we see that

$$(2) \quad \frac{1}{1 + q^{dp^{N+1}}} = \frac{1}{1 + q^{dp^N}} \cdot \frac{1 + q^{dp^N}}{1 + q^{dp^{N+1}}}.$$

From (1) and (2) we derive

$$\begin{aligned} \sum_{i=0}^{p-1} \mu_{-q}(a + idp^N + dp^{N+1}\mathbb{Z}_p) &= (1+q) \frac{(-1)^a q^a}{1+q^{dp^{N+1}}} \sum_{i=0}^{p-1} q^{idp^N} (-1)^i \\ &= (1+q) \frac{(-1)^a q^a}{1+q^{dp^N}} \cdot \frac{1+q^{dp^N}}{1+q^{dp^{N+1}}} \sum_{i=0}^{p-1} (-1)^i q^{idp^N} = \frac{(-1)^a q^a (1+q)}{1+q^{dp^N}} = \mu_{-q}(a + dp^N \mathbb{Z}_p). \end{aligned}$$

This distribution yields an integral for each non-negative integer  $m$  in the case  $d = 1$ ,

$$(3) \quad \int_{\mathbb{Z}_p} [a]_q^m d\mu_{-q}(a) = \lim_{N \rightarrow \infty} \sum_{a=0}^{p^N-1} [a]_q^m \frac{(-q)^a (1+q)}{1+q^{p^N}} = I_{-q}([a]_q^m),$$

which has a sense as we see readily that the limit is convergent.

Also, we easily see that (\*) is distribution on  $X$  for  $q \in \mathbb{C}_p$  with  $|1-q|_p < 1$ . We now define a  $q$ -Euler numbers  $E_{m,q} \in \mathbb{C}_p$  by making use of this integral:

$$I_{-q}([a]_q^m) = \int_{\mathbb{Z}_p} [a]_q^m d\mu_{-q}(a) = E_{m,q}.$$

Note that  $\lim_{q \rightarrow 1} E_{m,q} = E_m$ , where  $E_m$  are the  $m$ -th ordinary Euler numbers.

The generating function  $F_q(t)$  of  $E_{k,q}$ ,

$$(4) \quad F_q(t) = \sum_{k=0}^{\infty} E_{k,q} \frac{t^k}{k!},$$

is given by

$$(5) \quad F_q(t) = \lim_{\rho \rightarrow \infty} \frac{1}{[p^\rho]_{-q}} \sum_{i=0}^{p^\rho-1} (-q)^i e^{[i]_q t},$$

which satisfies the  $q$ -difference equation

$$(6) \quad F_q(t) = -qe^t F_q(qt) + 1.$$

If  $q = 1$  in Eq.(3)-(6), then we have

$$F_q(t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

Let  $\chi$  be a primitive Dirichlet character with conductor  $d(= \text{odd}) \in \mathbb{Z}_+$ , the set of natural numbers. Then we also define a generalized  $q$ -Euler numbers  $E_{n,\chi,q}$  as

$$E_{m,\chi,q} = \int_X \chi(a) [a]_q^m d\mu_{-q}(a) = \lim_{N \rightarrow \infty} \sum_{a=0}^{dp^N-1} [a]_q^m \chi(a) \frac{(-q)^a}{[dp^N]_{-q}},$$

where  $E_{m,\chi,q}$  are the  $m$ -th generalized  $q$ -Euler numbers attached to  $\chi$ .

The  $q$ -Euler polynomials in the variable  $x$  in  $\mathbb{C}_p$  with  $|x|_p \leq 1$  are defined by

$$\int_{\mathbb{Z}_p} [x+t]_q^n d\mu_{-q}(t) = E_{n,q}(x).$$

These can be written as

$$E_{n,q}(x) = \sum_{l=0}^n \binom{n}{l} q^{lx} E_{l,q} [x]_q^{n-l}.$$

Indeed we see

$$\int_{\mathbb{Z}_p} [x+t]_q^n d\mu_{-q}(t) = \sum_{k=0}^n \binom{n}{k} [x]_q^{n-k} q^{kx} \int_{\mathbb{Z}_p} [t]_q^k d\mu_{-q}(t).$$

For the integral  $I_{-q}$  we first see:

**Theorem 1.** For  $m \geq 0$ ,  $q \in \mathbb{C}_p$  with  $|1-q|_p < p^{-\frac{1}{p-1}}$ , we have

$$E_{m,q} = \int_{\mathbb{Z}_p} [x]_q^m d\mu_{-q}(x) = [2]_q \left( \frac{1}{1-q} \right)^m \sum_{k=0}^m \binom{m}{k} (-1)^k \frac{1}{1+q^{k+1}}.$$

*Proof.* We see

$$\begin{aligned} \frac{1+q}{1+q^{p^N}} \sum_{a=0}^{p^N-1} [a]_q^m q^a (-1)^a &= \frac{1+q}{1+q^{p^N}} \frac{1}{(1-q)^m} \sum_{a=0}^{p^N-1} (-1)^a q^a \sum_{j=0}^m \binom{m}{j} (-1)^j q^{aj} \\ &= \frac{1}{(1-q)^m} \frac{1+q}{1+q^{p^N}} \sum_{j=0}^m \binom{m}{j} (-1)^j \frac{1+q^{(j+1)p^N}}{1+q^{j+1}}. \end{aligned}$$

Since  $\lim_{N \rightarrow \infty} q^{p^N} = 1$  for  $|1-q|_p < p^{-\frac{1}{p-1}}$ , our assertion follows.

**Lemma 2.** For  $q \in \mathbb{C}_p$  with  $|1-q|_p < 1$ , we have  $\lim_{N \rightarrow \infty} q^{p^N} = 1$ .

Since

$$q^{p^N} = (q-1+1)^{p^N} = \sum_{l=0}^{p^N} \binom{p^N}{l} (q-1)^l.$$

From Theorem 1 and Lemma 2, we derive the following:

**Corollary 3.** For  $q \in \mathbb{C}_p$  with  $|1 - q|_p < 1$ ,  $m \in \mathbb{Z}_+$ , we have

$$\int_{\mathbb{Z}_p} [x]_q^m d\mu_{-q}(x) = [2]_q \left( \frac{1}{1 - q} \right)^m \sum_{k=0}^m \binom{m}{k} (-1)^k \frac{1}{1 + q^{k+1}}.$$

For  $f \in UD(\mathbb{Z}_p)$ , let us start with expression

$$\frac{1}{[p^N]_{-q}} \sum_{0 \leq j < p^N} (-q)^j f(j) = \sum_{0 \leq j < p^N} f(j) \mu_{-q}(j + p^N \mathbb{Z}_p),$$

representing  $q$ -analogue of Riemann sums for  $f$ .

The fermionic  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  will be defined as limit ( $N \rightarrow \infty$ ) of these sums, when it exists. A fermionic  $p$ -adic  $q$ -integral of function  $f \in UD(\mathbb{Z}_p)$  on  $\mathbb{Z}_p$  is defined as

$$(7) \quad I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x) q^x (-1)^x.$$

Note that if  $f_n \rightarrow f$  in  $UD(\mathbb{Z}_p)$ ; then

$$\int_{\mathbb{Z}_p} f_n d\mu_{-q}(x) \rightarrow \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x).$$

From (7), we derive the following theorem:

**Theorem 2.** For  $f \in UD(\mathbb{Z}_p)$ , we have

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0),$$

where  $f_1(x)$  is translation with  $f_1(x) = f(x + 1)$ .

Remark. In [1],  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{\infty} f(x) q^x.$$

For  $q = 1$  in Eq. (7), we have

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^x.$$

In view of notation,  $I_{-1}(f)$  can be written symbolically as  $I_{-1}(f) = \lim_{q \rightarrow -1} I_q(f)$ .

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